Assignment 7

Hand in no. 2, 3, 5, and 6 by November 2.

1. Consider maps from $\mathbb{R}$ to itself. Provide explicit examples of continuous maps with exactly one, two and three fixed points, and one map satisfying $|f(x) - f(y)| < |x - y|$ but no fixed points.

2. Let T be a continuous map on the complete metric space X. Suppose that for some k, T^k becomes a contraction. Show that T admits a unique fixed point. This generalizes the contraction mapping principle in the case $k = 1$.

3. Show that the equation $2x \sin x - x^4 + x = 0.001$ has a root near $x = 0$.

4. Study the solvability of
$$\sin^2 \pi x + 2x^2 = 2.0012 \ .$$
Hint: Consider $\Phi(1) = 2$ and shift things to the origin as done in Notes.

5. Can you solve the system of equations
$$x + y^4 = 0, \quad y - x^2 = 0.015 \ ?$$

6. Can you solve the system of equations
$$x + y - x^2 = 0, \quad x - y + xy \sin y = -0.005 \ ?$$

7. Show that the integral equation
$$y(x) = \alpha e^x - \int_0^1 \frac{\sin xt}{3 - t} \, y^3(t) \, dt \ ,$$
is solvable for sufficiently small α. Give an estimate on the smallness of α.

8. Let $A = (a_{ij})$ be an $n \times n$ matrix. Show that the matrix $I + A$ is invertible if $\sum_{i,j} a_{ij}^2 < 1$. Give an example showing that $I + A$ could become singular when $\sum_{i,j} a_{ij}^2 = 1$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0, f'(x_0) \neq 0$. Show that there exists some $\rho > 0$ such that
$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),$$
is a contraction. This provides a justification for Newton's method in finding roots for an equation.

10. Consider the iteration
$$x_{n+1} = \alpha x_n (1 - x_n), \ x_0 \in [0, 1] \ .$$
Find

   (a) The range of α so that $\{x_n\}$ remains in $[0, 1]$.
   (b) The range of α so that the iteration has a unique fixed point 0 in $[0, 1]$.
   (c) Show that for $\alpha \in [0, 1]$ the fixed point 0 is attracting in the sense: $x_n \to 0$ whenever $x_0 \in [0, 1]$.

11. Show that every continuous function from $[0, 1]$ to itself admits a fixed point. Here we don't need it a contraction. Suggestion: Consider the sign of $g(x) = f(x) - x$ at $0, 1$ where f is the given function.